Double Scattering Corrections to High-Energy Diffraction Scattering from Deuterons*

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The eikonal method is used to estimate the double scattering corrections to the differential cross sections for high-energy diffraction scattering from deuterons. By inverting the Fourier-Bessel transform the correction is expressed in terms of the amplitudes for scattering from free nucleons. Numerical estimates are made for the case of proton-deuteron scattering by assuming simple forms for the nucleon-nucleon scattering amplitudes and the deuteron form factor: Corrections to the differential cross sections are about -6% in the forward direction and increase in magnitude with increasing momentum transfer.

I. INTRODUCTION

BECAUSE of the difficulty of performing experiments with neutrons beams,[1] considerable work has been done with the aim of obtaining neutron cross sections from experiments using deuteron targets. Since the deuteron is loosely bound, it does indeed provide what is close to a free neutron target, but this simple picture is, unfortunately, good only up to a point. Several authors[2–4] have used the impulse approximation as a basis for a more reliable description of scattering from deuterons, while Glauber[5,6] has used the eikonal method to estimate the effects of double or "shadow" scattering upon the total cross section at high energies.

In this paper we discuss the extension of the work of Glauber to differential cross sections at energies high enough that the free-neutron and -proton cross sections are dominated by a narrow diffraction peak in the forward direction.[7] Under these conditions, the Fourier-Bessel transform used by Glauber can be inverted[8,9] and the amplitude for scattering from deuterons expressed directly in terms of the free-nucleon amplitudes.[9a] The phases of these amplitudes are, of course, unknown except in the forward direction where they can be determined from experiments by the use of the optical theorem.[10] We expect elastic scattering amplitudes to become more and more nearly pure positive imaginary as the energy increases and more and more inelastic channels open, but there seems to be some evidence for nonnegligible real parts[10] even at the highest energies presently available. In any case, once the phases are settled—by theory, experiment, or conjecture—the free-neutron differential cross section near the forward direction can be calculated from the deuteron and proton differential cross sections using the relations which we obtain below.

The scheme of the paper is as follows: In Sec. II we define the amplitudes with which we work and state the kinematical conditions which we assume in order that subsequent approximations be valid. For completeness we give the arguments leading to the Glauber formula in Sec. III and, by using the inverse Fourier-Bessel transformation, eliminate the eikonal function in favor of the free-neutron and proton scattering amplitudes. In order to clarify the approximations contained in the resulting formula we give an alternative derivation, based upon the impulse approximation, in Sec. IV. Expressions for the total cross section and various differential cross sections which result from the formula obtained in III and IV are given in Sec. V, together with an estimate of the double scattering correction to the angular distribution of protons scattered elastically and quasielastically from deuterons. Finally, in Sec. VI, we discuss our results and suggest experimental and theoretical tests of our approximate formulas.

II. KINEMATICS

We shall consider processes in which a fast incident particle strikes a deuteron and suffers a small deflection, either leaving the deuteron bound or raising its internal energy slightly so that the neutron and proton emerge

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[1] See, however, the recent work of H. Palevsky and his coworkers: H. Palevsky, J. A. Moore, R. L. Stearns, H. R. Muether, R. J. Sutter, R. E. Chrien, A. P. Jain, and K. Otnes, Phys. Rev. Letters **9**, 509 (1962); H. R. Muether, R. L. Stearns, R. E. Chrien, J. L. Friedes, R. J. Sutter, K. Otnes, and H. Palevsky, Bull. Am. Phys. Soc. **9**, 94 (1964); J. L. Friedes, R. E. Chrien, S. Mughabghab, V. W. Myers, R. J. Sutter, and H. Palevsky, *ibid.* **9**, 94 (1964).

[2] A. Everett, Phys. Rev. **126**, 831 (1962).

[3] A. Cromer, Phys. Rev. **129**, 1680 (1963).

[4] H. N. Pendleton, Phys. Rev. **131**, 1883 (1963). References to earlier papers can be found here and in Refs. 2 and 3.

[5] R. J. Glauber, Phys. Rev. **100**, 242 (1955).

[6] R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), p. 315.

[7] The narrow forward diffraction peak seems to be a universal property of cross sections at high energy. See, for example, the series of experiments reported by K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **10**, 376, 543 (1963); **11**, 425, 503 (1963).

[8] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

[9] B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962).

[9a] *Note added in proof*. This result has previously been obtained by V. Franco and R. J. Glauber. See R. J. Glauber, *International Conference on Nuclear Forces and the Few-Nucleon Problem* (Pergamon Press, Inc., New York, 1960), Vol. I, p. 233; V. Franco and R. J. Glauber, Bull. Am. Phys. Soc. **8**, 366 (1963); and V. Franco, thesis, Harvard University (unpublished). The author would like to thank Professor Glauber for calling this work to his attention.

[10] See the third paper cited in Ref. 7. This paper contains many references to earlier work.

in a low-energy scattering state. To be more specific, we shall consider the amplitudes

$$\langle \mathbf{p}'; \mathbf{q}_1', \mathbf{q}_2' | T | \mathbf{q}, \mathbf{P} \rangle$$

or

$$\langle \mathbf{p}'; \mathbf{P}' | T | \mathbf{p}, \mathbf{P} \rangle ,$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final momenta of the fast incident particle, $\mathbf{P}$ and $\mathbf{P}'$ are the initial and final (for elastic scattering) deuteron momenta, and $\mathbf{q}_1'$ and $\mathbf{q}_2'$ are the final proton and neutron momenta (for the inelastic case). We can treat the elastic and inelastic cases simultaneously by writing the amplitudes as

$$\langle \mathbf{p}'; \mathbf{P}', f | T | \mathbf{p}; \mathbf{P} \rangle ,$$

where $\mathbf{P}' = \mathbf{q}_1' + \mathbf{q}_2'$ in the inelastic case. The index f determines whether the final state contains a deuteron or a neutron and proton in an outgoing scattering state with relative momentum $2\mathbf{k} \equiv \mathbf{q}_1' - \mathbf{q}_2'$.

To simplify our calculations we shall ignore the spins of all the particles involved. Our results may still have some relation to reality, however, since it is likely that any spin dependence of the forces between particles tends to vanish at high energies[11]: The diffraction peak must be composed of particles coherent with the incident beam, and thus, in particular, having the same spin.

We shall generally work in the laboratory system, where $\mathbf{P} = 0$ and $\mathbf{P}' = -\mathbf{\Delta} \equiv \mathbf{p} - \mathbf{p}'$. The approximations we shall make will depend upon the following conditions:

$$P', k \ll m , \tag{1}$$

$$\Delta \ll p \approx p' , \tag{2}$$

and

$$ap \gg 1 , \tag{3}$$

where m is the nucleon mass and a is the range of the force between the incident particle and the nucleons. In other words, we require that the neutron-proton system be nonrelativistic and that the incident particle suffers only small changes in direction and energy during the collision. Because high-energy elastic scattering seems to be universally dominated by a diffraction peak, it seems reasonable to hope that a large fraction of events will satisfy these conditions, provided the incident energy is high enough. For these events the conservation of energy requires

$$E(p) - E(p') \approx \epsilon + (4m)^{-1}\Delta^2 + m^{-1}k^2 , \tag{4}$$

where $E(p) = (m^2 + p^2)^{1/2}$ and $\epsilon = 2m - m_d$ is the deuteron binding energy. This relation can be used to show that, under the conditions (1)–(3),

$$\mathbf{p} \cdot \mathbf{\Delta} \ll p\Delta , \tag{5}$$

so that $\mathbf{\Delta}$ is nearly perpendicular to $\mathbf{p}$.

Since we shall work in the laboratory system it is convenient to define an amplitude $F_f(p, \mathbf{\Delta})$, given by

$$F_f(p, \mathbf{\Delta}) = (4\pi)^{-1}(2E(p)2E(p'))^{1/2} \times \langle \mathbf{p}'; \mathbf{\Delta}, f | T | \mathbf{p}; 0 \rangle , \tag{6}$$

such that the differential cross section in the laboratory is simply

$$d\sigma_f(p, \mathbf{\Delta})/d\Omega(\mathbf{p}') = |F_f(p, \mathbf{\Delta})|^2 . \tag{7}$$

Of course, if the index f refers to a neutron-proton scattering state, it would be more precise to write

$$d\sigma_{\mathbf{k}}(p, \mathbf{\Delta})/d\Omega(\mathbf{p}')d^3k = (2\pi)^{-3}|F_{\mathbf{k}}(p, \mathbf{\Delta})|^2 . \tag{8}$$

Our aim is to write the amplitude $F_f(p, \mathbf{\Delta})$ in terms of the two-particle amplitudes

$$\langle \mathbf{p}', \mathbf{q}_i' | T | \mathbf{p}, \mathbf{q}_i \rangle ,$$

for the elastic scattering from neutrons ($i = n$) and protons ($i = p$). It is again convenient to introduce "laboratory" amplitudes $f_i(p, \Delta)$:

$$f_i(p, \Delta) = (4\pi)^{-1}(2E(p)2E(p'))^{1/2}\langle \mathbf{p}', \mathbf{q}_i' | T | \mathbf{p}, \mathbf{q}_i \rangle , \tag{9}$$

with the laboratory differential cross section given by

$$d\sigma_i(p, \Delta)/d\Omega(\mathbf{p}') = |f_i(p, \Delta)|^2 . \tag{10}$$

Finally, for future reference, we point out that, in terms of $f_i(p, \Delta)$ and $F_f(p, \mathbf{\Delta})$, the optical theorem reads

$$\mathrm{Im} f_i(p, 0) = (4\pi)^{-1} p \sigma_{i,T}(p)$$

and

$$\mathrm{Im} F_d(p, 0) = (4\pi)^{-1} p \sigma_T(p) , \tag{11}$$

where $\sigma_{i,T}(p)$ and $\sigma_T(p)$ are the total cross sections for scattering from free nucleons and deuterons, respectively.

III. EIKONAL METHOD

Assuming the two-particle differential cross sections to be dominated by a narrow forward peak, we can use the well-known Fourier-Bessel representation for the two-particle amplitudes[5,6,8,9]:

$$f_i(p, \Delta) = 2p \int_0^\infty b\, db\, J_0(\Delta b) \Gamma_i(p, b)$$

$$= (2\pi)^{-1} 2p \int d^2b \exp(i\mathbf{b} \cdot \mathbf{\Delta}) \Gamma_i(p, b) , \tag{12}$$

where $\Gamma_i(p, b)$ is given by the inverse transformation

$$\Gamma_i(p, b) = (2p)^{-1} \int_0^\infty \Delta\, d\Delta\, J_0(\Delta b) f_i(p, \Delta)$$

$$= (2\pi)^{-1}(2p)^{-1} \int d^2\Delta \exp(-i\mathbf{b} \cdot \mathbf{\Delta}) f_i(p, \Delta) . \tag{13}$$

In (12) and (13) b is the impact parameter: The vector $\mathbf{b}$ lies in a plane perpendicular to $\frac{1}{2}(\mathbf{p} + \mathbf{p}')$ and may be pictured as the vector from the origin of the potential to the classical trajectory of the scattered particle.

[11] H. Steiner, J. Arens, F. Betz, O. Chamberlain, B. Dieterle, P. Grannis, M. Hansroul, C. Schultz, G. Shapiro, L. Van Rossum, and D. Weldon, Bull. Am. Phys. Soc. 9, 95 (1964).

Insofar as the classical picture is valid, each value of b corresponds to the angular momentum

$$l(p,b)=p_{CM}b-\tfrac{1}{2}. \tag{14}$$

We have chosen the normalization of $\Gamma_i(p,b)$ such that

$$\Gamma_i(p,b)=(2i)^{-1}[\exp(i\chi_i(p,b))-1], \tag{15}$$

where

$$\chi_i(p,b)\approx 2\delta_i[p,l(p,b)] \tag{16}$$

is the eikonal function and $\delta_i(p,l)$ is the lth partial-wave phase shift (in general complex) for elastic scattering from the particle i. In a potential theory, under the conditions specified in Sec. II, $\chi_i(p,b)$ is just the integral of the potential along the classical path of the incident particle.[6]

In considering the scattering from a deuteron we shall neglect the motion of the neutron and proton during the brief period of interaction with the fast-moving incident particle, and assume them fixed at positions $\mathbf{r}_n=-\frac{1}{2}\mathbf{r}$ and $\mathbf{r}_p=\frac{1}{2}\mathbf{r}$, respectively, where $\mathbf{r}$ is their relative coordinate. Then, in analogy with the result obtained in potential theory,[6] we shall assume the scattering amplitude to be given by

$$F(p,\mathbf{\Delta};\mathbf{r})=(2\pi)^{-1}2p\int d^2b\,\exp(i\mathbf{b}\cdot\mathbf{\Delta})\Gamma(p,\mathbf{b};\mathbf{r}), \tag{17}$$

where

$$\begin{aligned}\Gamma(p,\mathbf{b};\mathbf{r})&=(2i)^{-1}[\exp(i\chi_n(p,b_n)+i\chi_p(p,b_p))-1]\\&=\Gamma_p(p,b_p)+\Gamma_n(p,b_n)+2i\Gamma_p(p,b_p)\\&\qquad\times\Gamma_n(p,b_n).\end{aligned} \tag{18}$$

In (18)

$$\mathbf{b}_p=\mathbf{b}-\tfrac{1}{2}\mathbf{r}_\perp$$

and

$$\mathbf{b}_n=\mathbf{b}+\tfrac{1}{2}\mathbf{r}_\perp, \tag{19}$$

where

$$\mathbf{r}_\perp=\mathbf{r}-p^{-2}(\mathbf{r}\cdot\mathbf{p})\mathbf{p} \tag{20}$$

is the projection of $\mathbf{r}$ onto a plane perpendicular to $\mathbf{p}$.

This expression for $F(p,\mathbf{\Delta};\mathbf{r})$, which follows from the assumption that the phase change produced by the neutron and proton when they are bound together in the deuteron is just the sum of the phase changes they would produce individually, would certainly be quite accurate in a potential theory if conditions (1)–(3) are satisfied. In a more realistic picture, however, the interactions between the incident particle and the nucleons are determined by the cloud of virtual particles surrounding the nucleons' cores, and these clouds might reasonably be expected to be distorted when the nucleons are brought together. If this distortion is significant one can hardly hope that (18) is correct. Because the deuteron has such an open structure, however, the nucleon clouds overlap only slightly, and (18) might well give a good approximation to $\Gamma(p,\mathbf{b};\mathbf{r})$.

To obtain the function $F_f(p,\mathbf{\Delta})$, defined in Sec. II, we must average $F(p,\mathbf{\Delta};\mathbf{r})$ over $\mathbf{r}$. Again guided by potential theory we shall make the reasonable assumption that

$$F_f(p,\mathbf{\Delta})=\int d^3r\psi_f^*(\mathbf{r})F(p,\mathbf{\Delta};\mathbf{r})\psi_d(\Omega), \tag{21}$$

where $\psi_f(\mathbf{r})$ is a two-nucleon wave function. From Eqs. (21), (18), (17), (15), and (13) we obtain, after some straightforward manipulation,

$$\begin{aligned}F_f(p,\mathbf{\Delta})=&\rho_f(\tfrac{1}{2}\mathbf{\Delta})f_p(p,\Delta)+\rho_f(-\tfrac{1}{2}\mathbf{\Delta})f_n(p,\Delta)\\&+i(2\pi p)^{-1}\int d^2\Delta'\rho_f(\mathbf{\Delta}')f_p(p,|\mathbf{\Delta}'+\tfrac{1}{2}\mathbf{\Delta}|)\\&\qquad\times f_n(p,|\mathbf{\Delta}'-\tfrac{1}{2}\mathbf{\Delta}|),\end{aligned} \tag{22}$$

where[12]

$$\rho_f(\mathbf{q})\equiv\int d^3r\psi_f^*(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})\psi_d(\Omega). \tag{23}$$

It should be noted that in (22) the interaction of the incident particle with the individual nucleons enters only via the scattering amplitudes $f_i(p,\Delta)$. Potential theory is used only to describe the low-energy neutron-proton system, and in this context it should be quite adequate.

IV. IMPULSE APPROXIMATION

While the argument given in Sec. III makes Eq. (22) rather plausible, it was based mainly upon the analogy with potential theory and does not make very clear just what has been left out of the approximate expression. It therefore seems useful to give an alternative "derivation" of (22) which makes somewhat clearer the nature of the approximation.

Because the deuteron is loosely bound it may be a good approximation to neglect the forces which hold it together (e.g., due to the exchange of pions) during the period of interaction with the incident particle. If so, we may use the impulse approximation to obtain the following expression for the amplitude $F_f(p,\mathbf{\Delta})$:

$$\begin{aligned}(4\pi)[2E(p)2E(p')]^{-1/2}F_f(p,\mathbf{\Delta})&=\langle\mathbf{p}';\mathbf{\Delta},f|T|\mathbf{p},0\rangle\\&\approx\sum_{\mathbf{q}_p',\mathbf{q}_n'}\sum_{\mathbf{q}_p,\mathbf{q}_n}\langle\mathbf{\Delta},f|\mathbf{q}_p',\mathbf{q}_n'\rangle\langle\mathbf{p}';\mathbf{q}_p',\mathbf{q}_n'|T|\mathbf{p};\mathbf{q}_p,\mathbf{q}_n\rangle\\&\qquad\times\langle\mathbf{q}_p,\mathbf{q}_n|0,d\rangle,\end{aligned} \tag{24}$$

where $|\mathbf{q}_p,\mathbf{q}_n\rangle$ is a state containing two free nucleons. The quantity $\langle\mathbf{q}_p,\mathbf{q}_n|\mathbf{P},f\rangle$ is simply related to φ_f, the two-nucleon wave function in momentum space for the state f:

$$\langle\mathbf{q}_p,\mathbf{q}_n|\mathbf{P},f\rangle=(2\pi)^3\delta^3(\mathbf{P}-\mathbf{q}_p-\mathbf{q}_n)\varphi_f[\tfrac{1}{2}(\mathbf{q}_p-\mathbf{q}_n)] \tag{25}$$

so that (24) may be written

$$\begin{aligned}F_f(p,\mathbf{\Delta})=&(4\pi)^{-1}[2E(p)2E(p')]^{1/2}\sum_{\mathbf{k}',\mathbf{k}}\varphi_f^*(\mathbf{k}')\\&\times\langle\mathbf{p}';\tfrac{1}{2}\mathbf{\Delta}+\mathbf{k}',\tfrac{1}{2}\mathbf{\Delta}-\mathbf{k}'|T|\mathbf{p};\mathbf{k},-\mathbf{k}\rangle\varphi_d(k).\end{aligned} \tag{26}$$

[12] The form factors $\rho_f(\mathbf{q})$ also occur in formulas describing the scattering of electrons and photons from deuterons. See, for example, L. Durand, Phys. Rev. **123**, 1393 (1961) and J. I. Friedman and H. W. Kendall, *ibid.* **129**, 2802 (1963).

Because the two-nucleon wave functions are large only for values of k and $k' \ll p$, the matrix element

$$\langle \mathbf{p}'; \mathbf{q}_p', \mathbf{q}_n' | T | \mathbf{p}; \mathbf{q}_p, \mathbf{q}_n \rangle$$

is required only when it is quite near the energy shell. We shall, in fact, assume that the amplitudes from which we construct it can be well approximated by their on-shell values.

To regain formula (22) we need take only those contributions to $\langle \mathbf{p}'; \mathbf{q}_p', \mathbf{q}_n' | T | \mathbf{p}, \mathbf{q}_p, \mathbf{q}_n \rangle$ which correspond to the diagrams shown in Fig. 1. The contribution from Fig. 1(a) is

$$(2\pi)^3 \delta^3(\mathbf{q}_n' - \mathbf{q}_n)(4\pi)[2E(p)2E(p')]^{-1/2} f_p(p,\Delta) \quad (27)$$

and that from Fig. 1(b) just (27) with the subscripts n and p interchanged. Diagrams 1(c) and 1(d) together give

$$(4\pi)^2[2\mathrm{E}(p)2E(p')]^{-1/2} f_p(p,\Delta_p) f_n(p,\Delta_n) \times[(\mu^2-(p+q_p-q_p')^2-i\epsilon)^{-1} +(\mu^2-(p+q_n-q_n')^2-i\epsilon)^{-1}], \quad (28)$$

where

$$\boldsymbol{\Delta}_p = \tfrac{1}{2}\boldsymbol{\Delta} + \mathbf{k} - \mathbf{k}', \quad \boldsymbol{\Delta}_n = \tfrac{1}{2}\boldsymbol{\Delta} - \mathbf{k} + \mathbf{k}', \quad (29)$$

and μ is the mass of the incident particle. To lowest order in the small momenta the first and second denominators in (28) are just $2\mathbf{p}\cdot(\mathbf{k}-\mathbf{k}')-i\epsilon$ and $-2\mathbf{p}\cdot(\mathbf{k}-\mathbf{k}')-i\epsilon$. Their sum is thus approximately $2\pi i\delta(2\mathbf{p}\cdot(\mathbf{k}-\mathbf{k}'))$, requiring that the vector $\mathbf{k}-\mathbf{k}'$ be perpendicular to $\mathbf{p}$.

Assembling our results we obtain

$$F_f(p,\boldsymbol{\Delta}) = \sum_{\mathbf{k},\mathbf{k}'} \varphi_f^*(\mathbf{k}')[(2\pi)^3\delta^3(\tfrac{1}{2}\boldsymbol{\Delta}+\mathbf{k}-\mathbf{k}') f_p(p,\Delta) + (2\pi)^3\delta^3(\tfrac{1}{2}\boldsymbol{\Delta}-\mathbf{k}+\mathbf{k}') f_n(p,\Delta) + 2\pi i\delta(2\mathbf{p}\cdot(\mathbf{k}-\mathbf{k}'))4\pi f_p(p,\Delta_p) f_n(p,\Delta_n)]\varphi_d(k). \quad (30)$$

Using

$$\sum_{\mathbf{k}} \varphi_f^*(\mathbf{k}-\mathbf{q})\varphi_d(\mathbf{k}) = \rho_f(\mathbf{q}), \quad (31)$$

it can easily be shown that this is equivalent to (22).

V. CROSS SECTIONS

The simplest result to be obtained from formula (22) is a consequence of the optical theorem. Using Eq. (11) we find

$$\sigma_T(p) = \sigma_{p,T}(p) + \sigma_{n,T}(p) + \mathrm{Re}\left[(2\pi p)^{-1}\int d^2\Delta' \rho_d(\Delta') f_p(p,\Delta') f_n(p,\Delta')\right], \quad (32)$$

which corresponds to the result obtained by Glauber.[5] If we follow him in assuming that $f_p(p,\Delta')$ and $f_n(p,\Delta')$ are pure positive imaginary and change slowly with Δ' compared to $\rho_d(\Delta')$, then the last term on the right-hand side of (32) gives the well-known "shadow" correction,

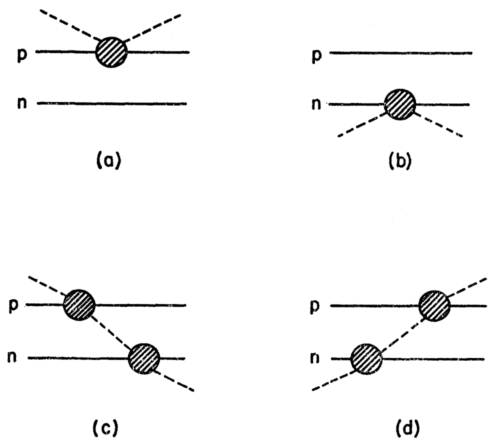

FIG. 1. Diagrams giving the single and double scattering contributions to the scattering from two free nucleons. The dashed line represents the incident particle, the solid lines the two nucleons.

$$-\sigma_{n,T}\sigma_{p,T}(4\pi)^{-1}\langle r^{-2}\rangle_d, \quad (33)$$

where

$$\langle r^{-2}\rangle_d = \int d^3x r^{-2}\psi_d^2(r). \quad (34)$$

For proton-deuteron above 1 BeV the correction, in this approximation, amounts to about 6% of σ_T, and its neglect would therefore produce an error of about 12% in a determination of $\sigma_{n,T}$ from σ_T and $\sigma_{p,T}$.

The differential cross section for elastic scattering is, of course, given by (7) with $f=d$:

$$d\sigma_d(p,\Delta)/d\Omega = |F_d(p,\Delta)|^2, \quad (35)$$

where

$$F_d(p,\Delta) = \rho_d(\tfrac{1}{2}\Delta)[f_p(p,\Delta)+f_n(p,\Delta)] + i(2\pi p)^{-1}\int d^2\Delta' \rho_d(\Delta') f_p(p, |\boldsymbol{\Delta}'+\tfrac{1}{2}\boldsymbol{\Delta}|) \times f_n(p, |\boldsymbol{\Delta}'-\tfrac{1}{2}\boldsymbol{\Delta}|). \quad (36)$$

Because the amplitudes $f_p(p,\Delta)$ and $f_n(p,\Delta)$ fall off more slowly with increasing Δ than does $\rho_d(\Delta)$ (at least in the energy range available for experiments at present), the relative importance of the correction term will increase with increasing Δ. Of course here, and below, we must remember that the range of Δ where we expect (22) to be valid is limited by condition (2).

The complete differential cross section for inelastic scattering is similarly obtained by substituting the expression for $F_{\mathbf{k}}(p,\boldsymbol{\Delta})$ given by (22) into Eq. (8). There seems little point in writing out this complicated expression here. A simpler and perhaps more useful expression can be obtained by integrating over the direction of the vector $\mathbf{k}$ and replacing the variable $|\mathbf{k}|$ by p', the momentum of the scattered particle. The resulting expression, which can be used to describe an experiment in which only the direction and momentum of the scattered particle are observed, is

$$d\sigma/d\Omega dp' = (2\pi)^{-3}(2E(p'))^{-1}mp'k\int d\Omega_{\mathbf{k}}|F_{\mathbf{k}}(p,\mathbf{\Delta})|^2$$

$$= (2\pi)^{-3}(2E(p'))^{-1}mp'k\Big\{|f_p(p,\Delta)|^2R(k;\tfrac{1}{2}\mathbf{\Delta},\tfrac{1}{2}\mathbf{\Delta})+\mathrm{Re}[f_p^*(p,\Delta)f_n(p,\Delta)R(k;\tfrac{1}{2}\mathbf{\Delta},-\tfrac{1}{2}\mathbf{\Delta})]$$

$$-2\,\mathrm{Im}\Big[f_p^*(p,\Delta)(2\pi p)^{-1}\int d^2\Delta' R(k;\tfrac{1}{2}\mathbf{\Delta},\mathbf{\Delta}')f_p(p,|\mathbf{\Delta}'+\tfrac{1}{2}\mathbf{\Delta}|)f_n(p,|\mathbf{\Delta}'-\tfrac{1}{2}\mathbf{\Delta}|)\Big]$$

$$+\tfrac{1}{2}(2\pi p)^{-2}\int d^2\Delta' d^2\Delta'' R(k;\mathbf{\Delta}',\mathbf{\Delta}'')f_p^*(p,|\mathbf{\Delta}'+\tfrac{1}{2}\mathbf{\Delta}|)f_n^*(p,|\mathbf{\Delta}'-\tfrac{1}{2}\mathbf{\Delta}|)$$

$$\times f_p(p,|\mathbf{\Delta}''+\tfrac{1}{2}\mathbf{\Delta}|)f_n(p,|\mathbf{\Delta}''-\tfrac{1}{2}\mathbf{\Delta}|)+\text{same with } n\leftrightarrow p\Big\}, \quad (37)$$

where the function[13] $R(k;\mathbf{a},\mathbf{b})$ is determined by the properties of the low-energy neutron proton system.

$$R(k;\mathbf{a},\mathbf{b})=\int d\Omega_{\mathbf{k}}\rho_{\mathbf{k}}^*(\mathbf{a})\rho_{\mathbf{k}}(\mathbf{b})$$

$$=4\pi\sum_l [2l+1)P_l(\hat{a}\cdot\hat{b})\rho_k^{(l)}(a)^*\rho_k^{(l)}(b), \quad (38)$$

with

$$\rho_k^{(l)}(a)=\frac{1}{2}\int_{-1}^{1} d(\hat{a}\cdot\hat{k})P_l(\hat{a}\cdot\hat{k})\rho_{\mathbf{k}}(\mathbf{a}). \quad (39)$$

The functions $\rho_k^{(l)}$, generalized to include the nucleon spin, have been studied extensively in connection with inelastic electron deuteron scattering[14]; many of these results can be directly applied to the present problem.

Under certain conditions a much simpler result can be obtained. Consider the quantity obtained by integrating (8) over d^3k and adding the elastic differential cross section:

$$\frac{d\sigma}{d\Omega}(p,\Delta)\equiv\int\frac{d^3k}{(2\pi)^3}|F_{\mathbf{k}}(p,\mathbf{\Delta})|^2+|F_d(p,\mathbf{\Delta})|^2, \quad (40)$$

applying to experiments in which only the direction of the scattered particle is measured. The kinematics then allow k to take a large range of values, only the smallest of which satisfy the conditions necessary for our approximations. We expect, however, that the amplitude $F\mathbf{k}(p,\mathbf{\Delta})$ will decrease rapidly for $k^2\gtrsim m\epsilon+\Delta^2$ because of the factors $\rho_{\mathbf{k}}(\mathbf{\Delta})$. (It should be remembered that, for small k and Δ, Δ is determined mainly by the direction of p' with only a weak dependence upon k.) If this rate of decrease is rapid enough we may, without serious error, extend the range of k integration to infinity and use (8) throughout this range. The closure relation for the neutron-proton states then gives

$$\int\frac{d^3k}{(2\pi)^3}\rho_{\mathbf{k}}^*(\mathbf{a})\rho_{\mathbf{k}}(\mathbf{b})+\rho_d^*(\mathbf{a})\rho_d(\mathbf{b})=\rho_d(|\mathbf{a}-\mathbf{b}|), \quad (41)$$

so that (40) becomes

$$\frac{d\sigma}{d\Omega}(p,\Delta)=|f_p(p,\Delta)|^2+\mathrm{Re}[f_p^*(p,\Delta)f_n(p,\Delta)]\rho_d(\Delta)$$

$$-2\,\mathrm{Im}\Big[f_p^*(p,\Delta)(2\pi p)^{-1}\int d^2\Delta'\rho_d(|\mathbf{\Delta}'-\tfrac{1}{2}\mathbf{\Delta}|)f_p(p,|\mathbf{\Delta}'+\tfrac{1}{2}\mathbf{\Delta}|)f_n(p,|\mathbf{\Delta}'-\tfrac{1}{2}\mathbf{\Delta}|)\Big]$$

$$+\tfrac{1}{2}(2\pi p)^{-2}\int d^2\Delta' d^2\Delta''\rho_d(|\mathbf{\Delta}'-\mathbf{\Delta}''|)f_p^*(p,|\mathbf{\Delta}'+\tfrac{1}{2}\mathbf{\Delta}|)f_n^*(p,|\mathbf{\Delta}'-\tfrac{1}{2}\mathbf{\Delta}|)$$

$$\times f_p(p,|\mathbf{\Delta}''+\tfrac{1}{2}\mathbf{\Delta}|)f_n(p,|\mathbf{\Delta}''-\tfrac{1}{2}\mathbf{\Delta}|)+\text{same with } n\leftrightarrow p. \quad (42)$$

The main difficulty in experimental applications of this formula will be to ascertain that only those final states with the scattered particle, a neutron and a proton (perhaps bound), and no additional particles, are recorded.

To obtain numerical estimates of the double scattering correction in (32), (35), and (42), the functions $f_p(p,\Delta)$, $f_n(p,\Delta)$, and $\rho_d(\Delta)$ are needed. For this

[13] The function $R(k;\mathbf{a},\mathbf{b})$ is essentially the Fourier transform of a space-time correlation function for the deuteron. See, for example, L. Van Hove, Phys. Rev. **95**, 249 (1954).

[14] See the first reference listed in 12.

purpose the forms

$$f_i(p,\Delta)=iA_i(p)\exp[-\alpha_i(p)\Delta^2] \qquad (43)$$

and

$$\rho_d(\Delta)=\exp(-\alpha_d\Delta^2) \qquad (44)$$

should be adequate, provided the parameters $A_i(p)$ and $\alpha_i(p)$ are chosen to fit the low-momentum transfer experimental data for scattering from free nucleons, and a reasonable value is used for α_d. For simplicity we shall assume below that $A_i(p)$ is a positive real number, which then must be proportional to the total cross section:

$$A_i(p)=(4\pi)^{-1}p\sigma_{i,T}(p)\,, \qquad (45)$$

but an imaginary part could be easily added. We shall use a value of 1.64 F^2 for α_d; as shown in Fig. 2 this gives a reasonable fit to $\rho_d(\Delta)$ in the region where it is large.

Using these simple approximate forms for the functions $f_i(p,\Delta)$ and $\rho_d(\Delta)$, the integrations in (32), (35),

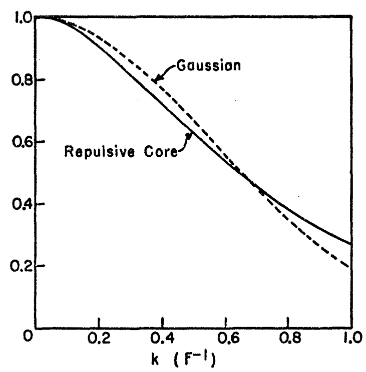

FIG. 2. Comparison of the repulsive core deuteron form factor, with the parameters given by Durand (see Ref. 12), with the Gaussian

$$\exp(-\alpha_d k^2),$$

with $\alpha_d=1.64$ F^2.

and (42) can be easily done, giving

$$\sigma_T(p)=\sigma_{p,T}(p)+\sigma_{n,T}(p)-(4\pi\alpha_t)^{-1}\sigma_{p,T}(p)\sigma_{n,T}(p)\,, \qquad (46)$$

$$\frac{d\sigma_d}{d\Omega}=\left\{\rho_d(\tfrac{1}{2}\Delta)\left[\left(\frac{d\sigma_p}{d\Omega}\right)^{1/2}+\left(\frac{d\sigma_n}{d\Omega}\right)^{1/2}\right]-\left(\frac{d\sigma_p}{d\Omega}\frac{d\sigma_n}{d\Omega}\right)^{1/4}(\sigma_{p,T}\sigma_{n,T})^{1/2}\times(8\pi\alpha_t)^{-1}\exp[-(4\alpha_t)^{-1}(\alpha_p-\alpha_n)^2\Delta^2]\right\}^2, \qquad (47)$$

and

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_p}{d\Omega}[1-\sigma_{n,T}(4\pi\alpha_t)^{-1}\exp(\alpha_t^{-1}\alpha_p^2\Delta^2)]+\frac{d\sigma_n}{d\Omega}[1-\sigma_{p,T}(4\pi\alpha_t)^{-1}\exp(\alpha_t^{-1}\alpha_n^2\Delta^2)]+\left(\frac{d\sigma_p}{d\Omega}\frac{d\sigma_n}{d\Omega}\right)^{1/2}[2\rho_d(\Delta)+\sigma_{p,T}\sigma_{n,T}(8\pi)^{-2}\times(\alpha_t^2-\alpha_d^2)^{-1}\exp((\alpha_p+\alpha_n)^{-1}2\alpha_p\alpha_n\Delta^2)], \qquad (48)$$

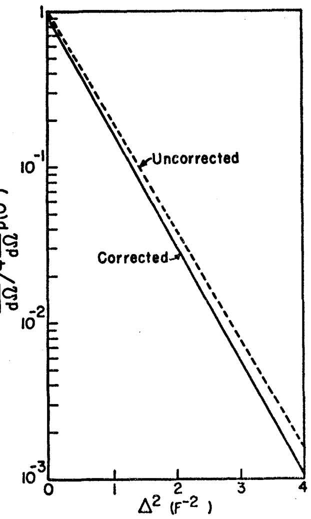

FIG. 3. The cross section for elastic proton deuteron scattering in the GeV range, with (solid line) and without (dashed line) double scattering corrections. Δ is the momentum transfer.

where

$$\alpha_t=\alpha_d+\alpha_p+\alpha_n\,. \qquad (49)$$

To illustrate these simplified formulas we choose parameters in rough agreement with presently available high-energy proton-proton and proton-neutron scattering data[7]:

$$\alpha_p=\alpha_n=0.4\ F^2\,,$$
$$\sigma_{p,T}=\sigma_{n,T}=40\ \text{mb} \qquad (50)$$

(that $\alpha_n=\alpha_p$ is, of course, a conjecture). We then find a double scattering correction to σ_T of -2.6 mb, or about -3%. This is somewhat smaller than Glauber's result, the difference arising because we here use somewhat smaller values for $\sigma_{p,T}$ and $\sigma_{n,T}$ and a different choice for $\rho_d(\Delta)$.

The double scattering corrections to the differential cross sections resulting from the parameters (50) are illustrated in Figs. 3 and 4. For elastic scattering the correction is about -6% in the forward direction and increases in magnitude with increasing Δ^2. (The curves at the larger values of Δ^2 should not be taken too

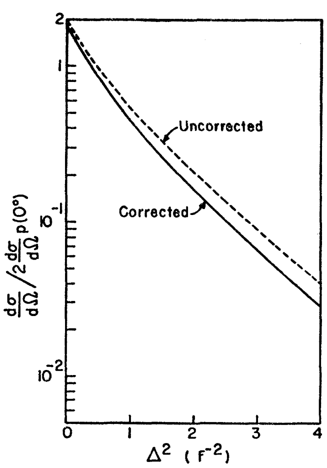

FIG. 4. The cross section for the sum of elastic and quasielastic proton-deuteron scattering in the GeV range with (solid line) and without (dashed line) double scattering corrections. Δ is the momentum transfer.

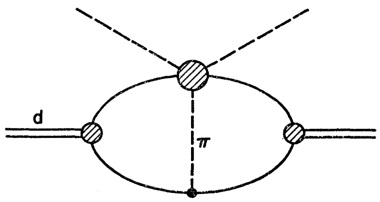

FIG. 5. A diagram which should be considered in investigating corrections to formula (22).

seriously since our approximate expression for the deuteron form factor is almost certainly too small there.) The integrated differential cross section, $d\sigma/d\Omega$, also has a correction of about -6% in the forward direction, and here too the magnitude of the relative correction increases with increasing Δ^2, but only up to $\Delta^2 \approx 5\ \mathrm{F}^{-2}$, where the last term in (48) begins to become significant.

VI. DISCUSSION

The results obtained above indicate that double scattering corrections to high-energy scattering from deuterons will be more important in the differential cross sections than in the total cross section, with larger effects at increasing momentum transfer. It is not difficult to understand this increase: If there is only a single scattering, with momentum transfer Δ, then the internal momentum of the neutron-proton system is changed by $\Delta/2$ and the free-particle scattering amplitude is that for momentum transfer Δ. For double scattering, on the other hand, the kinematics allow more freedom. Both the change in the internal momentum and the momentum transfer argument for the free-particle amplitudes can cover a range of values. In particular, the change in internal momentum can always take value in the neighborhood of zero, so that the deuteron has a good chance of remaining bound for arbitrarily large values of Δ. Furthermore, the momentum transfer arguments can both be in the neighborhood of $\Delta/2$, so that if we assume the form (43) for the free-particle amplitudes we have a factor $\exp-\Delta^2/2$ rather than the $\exp-\Delta^2$ which occurs in the single scattering term. Naturally, because of the nature of our approximations, we cannot use our results at large momentum transfers. They should serve to warn us, however, that the simple impulse approximation without multiple scattering corrections may be quite bad for large momentum transfers.

The approximations which we made above to obtain the double scattering corrections should be valid under the conditions (1)–(3). The argument leading to (22) is certainly not rigorous, though, and we would have more confidence in this result if it were verified experimentally. This is fortunately possible: As indicated in Glauber's paper,[5] if we assume a charge-symmetric pion-nucleon interaction then the pion-neutron amplitudes are determined by the pion-proton amplitudes. Apart from the problem of phase, therefore, we have all the input data required to calculate pion deuteron cross sections at high energy from (22); the results can then be checked by comparing with experimental results. It should be realized that a discrepancy does not necessarily invalidate our basic approach; there may be a strong spin dependence or a surprisingly rapid phase variation. These complications can, in principle, be included using the present approach, but more information on the free-particle scattering amplitude than is presently available would be required. We are further limited by our ignorance of the high-momentum components of the deuteron, but this should not produce serious errors as long as condition (1) is satisfied.

We can also try to check the validity of our approximations theoretically. It should be possible to estimate corrections to the impulse approximation such as those originating from the diagram shown in Fig. 5. In diagrammatic calculations such as this, however, the low-energy and high-energy aspects of the problem, which we thus far have kept separate, apparently become thoroughly mixed. For example, the pion in Fig. 5 can be regarded as either the result of production in the high-energy collision or as one of the pions which are occasionally exchanged between the nucleons, producing the binding. It therefore seems that for high-energy collisions corrections of this sort can be treated properly only by starting from a fully relativistic theory. In analogy with similar diagrams contributing to the electromagnetic form factor of the deuteron[15] one expects a certain part of the contribution from the diagram of Fig. 5 to be accounted for by a good deuteron wave function; the difficulty is to separate this part from the remainder which constitutes a correction to the impulse approximation.

In conclusion we should like to emphasize the usefulness of experimental studies of high-energy elastic and quasielastic scattering from deuterium. These experiments have the potential of furnishing cross sections for scattering from neutrons and, in addition, may eventually provide a useful method for studying the deuteron, especially in the event the development of a relativistic theory of this bound state allows us to relax condition (1).

[15] F. Gross, Phys. Rev. **134**, B405 (1964).